

**The dynamical structure of four-dimensional
Chamseddine's gauge theory of gravity**

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We perform the Dirac hamiltonian analysis of a four-dimensional gauge theory of gravity with an action of topological type, which generalizes some well-known two-dimensional models. We show that, in contrast with the two-dimensional case, the theory has a non-vanishing number of dynamical degrees of freedom and that its structure is very similar to higher-dimensional Chern-Simons gravity.

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1. Introduction

The problem of finding a consistent quantum theory of gravity is one of the most fascinating challenges in today's theoretical physics. Among the various approaches to the problem, many attempts have been done to write down general relativity as a gauge theory, in the hope that what has been learned in the quantization of gauge theories can be exploited also in the case of gravitation. Some partial results have been obtained since the sixties in the direction of identifying GR with some kind of gauge theory of the Poincaré group [1]. However it has not been possible till now to do this in a completely satisfying way. A closer equivalence of gravity with gauge theories can however be obtained in lower dimensions [2-3]. It has been shown, in fact, that the 2- and 3-dimensional analogue of GR can be written down as vector gauge theories of the lower-dimensional (anti)-de Sitter or Poincaré group with action of topological kind, where by topological we mean that no background metric is introduced in the formalism.

More precisely, in three dimensions one adopts a Chern-Simons action for the (anti)-de Sitter or Poincaré group [2]. After a suitable identification of the components of the gauge connection with the dreibeins and the Lorentz connection of the manifold, one recovers the three-dimensional Einstein-Hilbert action. An analogous mechanism works in two dimensions if the action is chosen to be of the BF type [3].

These lower-dimensional models have in common the absence of local degrees of freedom and this property renders their study much easier than higher-dimensional theories. In particular, one can perform their exact quantization in a straightforward way [2-3]. It is therefore natural to ask whether these models can be generalized to higher dimensions and if in this case they acquire dynamical degrees of freedom and if their quantization can still be easily achieved. Generalizations to higher dimensions are indeed possible, as was first shown in [4], but the resulting theories can no longer be identified with higher-dimensional general relativity.

The case of odd dimensions has been discussed in several papers [4-5]. The action is in this case the straightforward generalization to higher dimensions of the Chern-Simons action, and after the identification of the gauge fields with the geometrical quantities gives rise to a gravitational action which is a sum with given coefficients of Gauss-Bonnet

terms. (These generalized gravitational actions in higher dimensions were first introduced by Lovelock [6]).

The even-dimensional case is less trivial. One possibility is to consider higher dimensional BF theories: in this case the gauge fields must be coupled to higher rank tensor fields [7]. A second possibility was suggested by Chamseddine [4] and is perhaps closer in spirit to the two-dimensional model. According to his proposal, one proceeds as in two dimensions, and couples a scalar multiplet to the field strength of the relevant gauge group. After the usual identifications of gauge potentials and geometric quantities, one still obtains an action which is a sum of Euler densities, which however are now coupled to the scalar fields. In addition, the action includes some further terms involving products of curvature and torsion. Some of the physical implications of these models have been discussed in [8].

We again remark that in general these higher dimensional models are quite different from GR, especially in the Poincaré case, where only the highest order Gauss-Bonnet term survives and hence no term proportional to the Einstein-Hilbert action is included.

We have recalled that in three dimensions the Chern-Simons action for gravity possesses no degrees of freedom. This is not true however for its higher dimensional generalizations, as was shown in [9]. Analogously, the two-dimensional Chamseddine's lagrangian for gravitation does not have dynamical degrees of freedom. It would be interesting to know if this property extends to higher dimensions. Investigations based on a perturbative expansion have shown that this may depend on the specific background chosen for the calculation [8]. A deeper understanding of the phase space of the model should however be obtained by using the hamiltonian formalism. This is the purpose of the present paper.

As usual with generally covariant theories, we obtain a constrained hamiltonian system. The action is first-order in the time derivatives and the hamiltonian results to be a linear combination of the constraints. Adopting the Dirac procedure [10], we separate the constraints in first class and second class. The first class constraints generate the gauge transformations and spatial diffeomorphisms. Using standard methods [11], one can then calculate the number of dynamical degrees of freedom. Our analysis is simplified by the fact that the hamiltonian formulation of our model displays many similarities with that of

Chern-Simons theories, which was discussed some time ago in [9].

We shall consider only the case of $D = 4$, but the discussion can be easily extended to any even dimension. Also the generalization to more general gauge groups may be obtained in a straightforward way. We find that, like in the Chern-Simons theories, in higher dimensions the number of degrees of freedom does not vanish. A further analogy with the Chern-Simons theories is that the generator of the time-like diffeomorphisms is not independent from the other constraints. This is a good feature in view of the quantization of the model, since it is well known that usually this is the constraint that is most difficult to solve. On the other hand, the Dirac bracket structure is very involved in our case, and we were not able to compute it explicitly.

2. Gauge theories of gravity in $2n$ dimensions

In a even number of dimensions, it is not possible to define a Chern-Simons action. However, one can construct a different sort of action, which still does not depend on any background metric and may therefore be called "topological".

In a $2n$ -dimensional spacetime the most natural choices for the gauge group of gravitation are the Poincaré group $ISO(1, 2n - 1)$, the de Sitter group $SO(1, 2n)$ or the anti-de Sitter group $SO(2, 2n - 1)$, depending on the value of the cosmological constant λ , which takes the value $\lambda = 0$, $\lambda = 1$ or $\lambda = -1$ respectively. The last two groups admit as invariant tensor the totally antisymmetric $(2n + 1)$ -tensor $\epsilon_{A_1 \dots A_{2n+1}}$, but the $\lambda = 0$ case can be easily recovered by Inönü-Wigner contraction.

The generators M_{AB} of the gauge algebra satisfy the commutation relations

$$[M_{AB}, M_{CD}] = \frac{1}{2}(h_{AD}M_{BC} - h_{AC}M_{BD} - h_{BD}M_{AC} + h_{BC}M_{AD}) \quad (2.1)$$

with $h_{AB} = \text{diag}(-1, 1, \dots, 1, \lambda)$ and the group indices A, B, \dots run from 0 to $2n$.

As in standard Yang-Mills theory, local invariance under the gauge group can be enforced by introducing a gauge connection one-form A^{AB} with field strength 2-form $F^{AB} \equiv (dA + \frac{1}{2}[A, A])^{AB} = dA^{AB} + A^{AC}A^{CB}$, where the indices are summed by means of h_{AB} . A gauge-invariant action of topological form can then be constructed by taking the $2n$ -form given by the exterior product of n field strengths. However, in order to construct

a group invariant, one is forced to introduce a scalar multiplet η^A in the fundamental representation of the group[†]. The action can then be written as [4]

$$I = \int_{M_{2n}} \epsilon_{A_1 \dots A_{2n+1}} \eta^{A_1} F^{A_2 A_3} \dots F^{A_{2n} A_{2n+1}} \quad (2.2)$$

The field equations, obtained by varying the action with respect to the scalar and to the gauge potential are given respectively by:

$$\begin{aligned} \epsilon_{A_1 \dots A_{2n+1}} F^{A_2 A_3} \dots F^{A_{2n} A_{2n+1}} &= 0 \\ \epsilon_{A_1 \dots A_{2n+1}} F^{A_2 A_3} \dots F^{A_{2n-2} A_{2n-1}} D\eta^{A_1} &= 0 \end{aligned} \quad (2.3)$$

where D is the gauge covariant derivative.

In order to establish a relation with $2n$ -dimensional gravity, one can now make the identifications $A^{ab} = \omega^{ab}$, $A^{a,2n} = e^a$, where $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ and $e^a = e^a_\mu dx^\mu$ are the spin connection and vielbein 1-forms of the $2n$ -dimensional manifold and the indices $a, b, \dots = 0, \dots, 2n-1$ refer to the Lorentz subgroup $SO(1, 2n-1)$ of the gauge group. It follows that $F^{ab} = R^{ab} + \lambda e^a e^b$, $F^{a,2n} = T^a$, where R^{ab} and T^a are the curvature and the torsion 2-forms of the $2n$ -dimensional manifold, which are defined respectively as

$$R^{ab} = d\omega^{ab} + \omega^{ac}\omega^{cb} \quad T^a = de^a + \omega^{ab}e^b \quad (2.4)$$

and satisfy the Bianchi identities

$$\begin{aligned} \nabla T^a &\equiv dT^a + \omega^{ab}T^b = R^{ab}e^b \\ \nabla R^{ab} &\equiv (dR + \omega R - R\omega)^{ab} = 0 \end{aligned} \quad (2.5)$$

∇ being the Lorentz covariant derivative on the spacetime. The scalar field η^A is also decomposed in a Lorentz scalar $\eta \equiv \eta^{2n}$ and a Lorentz vector η^a .

With these identifications, the action (2.2) becomes

$$\begin{aligned} I = \int_{M_{2n}} &\left[\epsilon_{a_1 \dots a_{2n}} \eta (R^{a_1 a_2} + \lambda e^{a_1} e^{a_2}) \dots (R^{a_{2n-1} a_{2n}} + \lambda e^{a_{2n-1}} e^{a_{2n}}) \right. \\ &\left. + 2n \epsilon_{a_1 \dots a_{2n}} \eta^{a_1} T^{a_2} (R^{a_3 a_4} + \lambda e^{a_3} e^{a_4}) \dots (R^{a_{2n-1} a_{2n}} + \lambda e^{a_{2n-1}} e^{a_{2n}}) \right] \end{aligned} \quad (2.6)$$

[†] A different sort of generalization with an action linear in the F^{AB} can be obtained by introducing higher-rank forms [7].

The first term in the action (2.6) represents a sum of Gauss-Bonnet forms multiplied by the scalar field η [5]. In the $\lambda = 0$ case only the term $\eta \epsilon_{a_1 \dots a_{2n}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}}$ in the integral survives. This is the Euler class of the manifold and in the absence of the scalar field, it would be a total derivative.

In geometric form, the field equations (2.3) can be written as

$$\begin{aligned} \epsilon_{a_1 \dots a_{2n}} (R^{a_1 a_2} + \lambda e^{a_1} e^{a_2}) \dots (R^{a_{2n-1} a_{2n}} + \lambda e^{a_{2n-1}} e^{a_{2n}}) &= 0 \\ \epsilon_{a_1 \dots a_{2n}} T^{a_2} (R^{a_3 a_4} + \lambda e^{a_3} e^{a_4}) \dots (R^{a_{2n-1} a_{2n}} + \lambda e^{a_{2n-1}} e^{a_{2n}}) &= 0 \\ \epsilon_{a_1 \dots a_{2n}} (R^{a_2 a_3} + \lambda e^{a_2} e^{a_3}) \dots (R^{a_{2n-2} a_{2n-1}} + \lambda e^{a_{2n-2}} e^{a_{2n-1}}) (\nabla \eta^{a_1} + \lambda \eta e^{a_1}) &= 0 \\ \epsilon_{a_1 \dots a_{2n}} [2T^{a_2} (R^{a_3 a_4} + \lambda e^{a_3} e^{a_4}) \dots (R^{a_{2n-3} a_{2n-2}} + \lambda e^{a_{2n-3}} e^{a_{2n-2}}) (\nabla \eta^{a_1} + \lambda \eta e^{a_1}) \\ + (R^{a_1 a_2} + \lambda e^{a_1} e^{a_2}) \dots (R^{a_{2n-3} a_{2n-2}} + \lambda e^{a_{2n-3}} e^{a_{2n-2}}) (\nabla \eta - \eta^b e_b)] &= 0 \end{aligned}$$

3. The 4-dimensional theory

In the following, we shall concentrate our attention on four dimensions. The gauge group is given in this case by $ISO(1, 3)$, $SO(1, 4)$ or $SO(2, 3)$ with generators M_{AB} , $A, B = 0, \dots, 4$ and commutation relations (2.1), where now $h_{AB} = \text{diag}(-1, 1, 1, 1, \lambda)$.

The components of the gauge field strength are given by

$$F_{\mu\nu}^{AB} = \partial_\mu A_\nu^{AB} - \partial_\nu A_\mu^{AB} + A_\mu^{AC} A_\nu^{CB} - A_\nu^{AC} A_\mu^{CB} \quad (3.1)$$

where A_μ^{AB} ($\mu = 0, \dots, 3$) is the gauge connection.

The four-dimensional Chamseddine action can be written as

$$I = \int_{M_4} d^4x \epsilon_{ABCDE} \epsilon^{\mu\nu\rho\sigma} \eta^A F_{\mu\nu}^{BC} F_{\rho\sigma}^{DE} \quad (3.2)$$

This action is invariant, up to boundary terms, under the standard gauge transformations with parameter χ^{AB} :

$$\delta_G A_\mu^{AB} = -D_\mu \chi^{AB} \quad \delta_G \eta^A = \chi^{AB} \eta^B \quad (3.3)$$

where D_μ is the gauge covariant derivative. The action is also invariant under diffeomorphisms of the spacetime manifold M with parameter ξ^μ :

$$\delta_D A_\mu^{AB} = \mathcal{L}_\xi A_\mu^{AB} = \xi^\nu \partial_\nu A_\mu^{AB} + A_\nu^{AB} \partial_\mu \xi^\nu \quad \delta_D \eta^A = \mathcal{L}_\xi \eta^A = \xi^\mu \partial_\mu \eta^A \quad (3.4)$$

where \mathcal{L}_ξ is the Lie derivative in the direction of ξ^μ . It is useful to define improved diffeomorphisms [12], that differ from (3.4) by a gauge transformation with parameter $\chi^{AB} = \xi^\nu A_\nu^{AB}$. One has

$$\delta_I A_\mu^{AB} = \xi^\nu F_{\nu\mu}^{AB} \quad \delta_I \eta^A = \xi^\mu \partial_\mu \eta^A + \xi^\mu A_\mu^{AB} \eta^B = \xi^\mu D_\mu \eta^A \quad (3.5)$$

Varying the action (3.2) with respect to η^A and A_μ^{AB} one obtains the field equations

$$\begin{aligned} \epsilon_{ABCDE} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{BC} F_{\rho\sigma}^{DE} &= 0 \\ \epsilon_{ABCDE} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{BC} D_\rho \eta^A &= 0 \end{aligned} \quad (3.6)$$

In the following we shall make repeated use of the Bianchi identities

$$D_{[\nu} F_{\rho\sigma]} = \epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0 \quad (3.7)$$

In order to perform the Hamiltonian analysis, we assume that the spacetime manifold has topology $R \times \Sigma$ and decompose the 1-form A^{AB} as

$$A_\mu^{AB} dx^\mu = A_0^{AB} dt + A_i^{AB} dx^i \quad (i = 1, 2, 3)$$

The action can then be decomposed as

$$I = \int_R \int_\Sigma dt d^3x \left[l_{AB}^i(\eta, A) \dot{A}_i^{AB} + A_0^{AB} K_{AB}(\eta, A) \right] \quad (3.8)$$

with

$$l_{AB}^i = \epsilon_{ABCDE} \epsilon^{ijk} F_{jk}^{CD} \eta^E \quad (3.9)$$

$$K_{AB} = \epsilon_{ABCDE} \epsilon^{ijk} F_{jk}^{CD} D_i \eta^E = D_i l_{AB}^i \quad (3.10)$$

The field equations (3.6) are decomposed accordingly as follows:

$$\epsilon_{ABCDE} \epsilon^{ijk} (\dot{A}_i^{BC} - D_i A_0^{BC}) F_{jk}^{DE} = 0 \quad (3.11)$$

$$\epsilon_{ABCDE} \epsilon^{ijk} (2\dot{A}_j^{CD} D_k \eta^E + F_{jk}^{CD} \dot{\eta}^E - 2D_j A_0^{CD} D_k \eta^E + F_{jk}^{CD} A_0^{EF} \eta^F) = 0 \quad (3.12)$$

$$K_{AB} = 0 \quad (3.13)$$

which follow from the variation of η^A , A_i^{AB} , A_0^{AB} respectively. The last equation can be interpreted as a constraint, with the non-dynamical fields A_0^{AB} playing the role of Lagrange multipliers.

4. Analysis of the constraints

The action (3.8) is first order in the time derivatives. The hamiltonian analysis for first-order actions can be performed either using the formalism introduced by Fadeev and Jackiw [13], which postulates an unusual type of brackets in configuration space, or by means of the standard Dirac method for constrained systems [10]. Due to the presence of second class constraints, it is more convenient in our case to make recourse to the Dirac formalism (see ref. [14] for a general discussion). Therefore, since the action is linear in the time derivatives of A_i^{AB} and does not contain the time derivatives of the η^A , in addition to the K_{AB} , we must impose the 35 constraints

$$\begin{aligned}\phi_{AB}^i &\equiv p_{AB}^i - l_{AB}^i \approx 0 \\ \psi_A &\equiv \pi_A \approx 0\end{aligned}\tag{4.1}$$

where ϕ_{AB}^i and π_A are the momenta canonically conjugate to A_i^{AB} and η^A respectively. The Poisson brackets between these constraints are

$$\begin{aligned}\Omega_{AB,CD}^{ij} &\equiv \{\phi_{AB}^i, \phi_{CD}^j\} = \frac{\delta l_{CD}^j}{\delta A_i^{AB}} - \frac{\delta l_{AB}^i}{\delta A_j^{CD}} = -2\epsilon_{ABCDE} \epsilon^{ijk} D_k \eta^E \\ \Omega_{AB,C}^i &\equiv \{\phi_{AB}^i, \psi_C\} = -\frac{\delta l_{AB}^i}{\delta \eta^C} = \epsilon_{ABCDE} \epsilon^{ijk} F_{jk}^{DE} \\ \Omega_{AB} &\equiv \{\psi_A, \psi_B\} = 0\end{aligned}\tag{4.2}$$

It is also convenient to replace the constraints K_{AB} with new constraints G_{AB} , which generate the gauge transformations (3.3):

$$-G_{AB} = K_{AB} + D_i \phi_{AB}^i + \eta_{[A} \psi_{B]} = D_i p_{AB}^i + \eta_{[A} \pi_{B]}$$

Indeed, it is easy to check that

$$\begin{aligned}\delta A_i^{AB} &= \left\{ A_i^{AB}, \int_{\Sigma} \chi^{CD} G^{CD} \right\} = -D_i \chi^{AB} \\ \delta \eta^A &= \left\{ \eta^A, \int_{\Sigma} \chi^{BC} G^{BC} \right\} = \chi^{AB} \eta^B\end{aligned}\tag{4.3}$$

One has

$$\begin{aligned}\{G_{AB}, G_{CD}\} &= \frac{1}{2}(h_{AD}G_{BC} - h_{AC}G_{BD} - h_{BD}G_{AC} + h_{BC}G_{AD}) \\ \{\phi_{AB}^i, G_{CD}\} &= \frac{1}{2}(h_{AD}\phi_{BC}^i - h_{AC}\phi_{BD}^i - h_{BD}\phi_{AC}^i + h_{BC}\phi_{AD}^i) \\ \{\psi_A, G_{BC}\} &= \frac{1}{2}(h_{AB}\psi_C - h_{AC}\psi_B)\end{aligned}\tag{4.4}$$

It follows that the G_{AB} are first class and their Poisson brackets reproduce the gauge algebra (2.1).

The Hamiltonian density is now

$$\mathcal{H} = A_0^{AB}G_{AB} + u_i^{AB}\phi_{AB}^i + v^A\psi_A\tag{4.5}$$

where u_i^{AB} and v^A are Lagrange multipliers enforcing the constraints ϕ_{AB}^i and ψ_A . The consistency condition $\dot{G}_{AB} \approx 0$ is automatically satisfied because the G_{AB} are first class, while the other consistency conditions

$$\begin{aligned}\dot{\phi}_{AB}^i &= \{\phi_{AB}^i, H\} \approx u_j^{CD}\Omega_{AB,CD}^{ij} + v^C\Omega_{AB,C}^i = 0 \\ \dot{\psi}_C &= \{\psi_C, H\} \approx u_i^{AB}\Omega_{AB,C}^i = 0\end{aligned}\tag{4.6}$$

give restrictions on the Lagrange multipliers u_i^{AB} , v^C . Hence, no further constraints appear.

We have already seen that the constraints G_{AB} are first class. In order to investigate the nature of the constraints ϕ_{AB}^i , ψ_B , one must consider the matrix $\Omega_{\alpha\beta}$ formed with the Poisson brackets of the constraints [14], where α, β stay for the indices a, b, i , etc.

$$\Omega_{\alpha\beta} = \begin{pmatrix} \{\phi_{AB}^i, \phi_{CD}^j\} & \{\phi_{AB}^i, \psi_F\} \\ \{\psi_E, \phi_{CD}^j\} & \{\psi_E, \psi_F\} \end{pmatrix}\tag{4.7}$$

It turns out that this matrix is not invertible on the constraint surface and therefore some combinations of the constraints ϕ_{AB}^i , ψ_A are first class. To show this, let us find the null eigenvectors V_β of $\Omega_{\alpha\beta}$, using the relations (4.2). One must solve the matrix equation

$$\Omega_{\alpha\beta}V_\beta = \begin{pmatrix} \Omega_{AB,CD}^{ij} & \Omega_{AB,F}^i \\ -\Omega_{CD,E}^j & 0 \end{pmatrix} \begin{pmatrix} V_j^{CD} \\ V^F \end{pmatrix} = 0\tag{4.8}$$

This yields

$$-\Omega_{CD,E}^j V_{(l)j}^{CD} = \epsilon_{ABCDE} \epsilon^{ijk} F_{jk}^{AB} V_{(l)i}^{CD} = 0\tag{4.9}$$

$$\Omega_{AB,CD}^{ij} V_{(l)j}^{CD} + \Omega_{AB,F}^i V_{(l)}^F = -\epsilon_{ABCDE} \epsilon^{ijk} [2V_{(l)j}^{CD} D_k \eta^E + F_{jk}^{CD} V_{(l)}^E] = 0\tag{4.10}$$

The first equation admits the three solutions $V_{(l)i}^{CD} = F_{li}^{CD}$, with $l = 1, 2, 3$. Substituting in the second, one gets $V_{(l)}^E = D_l \eta^E$. This is indeed a consequence of the identity

$$\epsilon_{ABCDE} \epsilon^{ijk} [2F_{lj}^{CD} D_k \eta^E + F_{jk}^{CD} D_l \eta^E] = \delta_l^i K_{AB} \approx 0 \quad (4.11)$$

Hence, the 35×35 matrix $\Omega_{\alpha\beta}$ has at least three null eigenvectors

$$V_{(l)\beta} = \begin{pmatrix} F_{li}^{CD} \\ D_l \eta^E \end{pmatrix} \quad (4.12)$$

which correspond to first class constraints. These are given explicitly by

$$H_l = \phi_{AB}^i F_{li}^{AB} + \psi_A D_l \eta^A = p_{AB}^i F_{li}^{AB} + \pi_A D_l \eta^A$$

and generate the improved spatial diffeomorphisms (3.5). In fact,

$$\delta A_i^{AB} = \left\{ A_i^{AB}, \int_{\Sigma} H_l \xi^l \right\} = \xi^l F_{li}^{AB} \quad \delta \eta^A = \left\{ \eta^A, \int_{\Sigma} H_l \xi^l \right\} = \xi^l D_l \eta^A \quad (4.13)$$

It is interesting to note that, in contrast with the two-dimensional case [2], these constraints are in general independent of the constraints G_{AB} , generating local gauge transformations. A similar situation arises in Chern-Simons theories, where the dependence occurs only in three dimensions, but not in higher dimensions [9].

Another important analogy with Chern-Simons theories is that the generator of time diffeomorphisms is a linear combination of the first class constraints G_{AB} and H_l , since this symmetry is not independent from the other ones. This can be proved by showing that on-shell the time diffeomorphisms can be written as space diffeomorphisms with suitable parameters. The proof goes essentially like in the Chern-Simons case [9]: indeed, the field equations (3.11), (3.12) can be written in terms of the matrix $\Omega_{\alpha\beta}$ as

$$\Omega_{\alpha\beta} F_{0\beta} = 0 \quad (4.14)$$

where

$$F_{0\beta} = \begin{pmatrix} F_{0i}^{CD} \\ D_0 \eta^E \end{pmatrix} \quad (4.15)$$

Hence, if $\Omega_{\alpha\beta}$ has only the three null eigenvectors $V_{(l)\beta}$, then some parameters ζ^l must exist such that $F_{0\beta} = \zeta^l V_{(l)\beta}$. Thus, for a time diffeomorphism, parametrized by $\xi^\mu = (\xi^0, 0)$, (3.5) can be written as

$$\delta_I \begin{pmatrix} A_i^{CD} \\ \eta^E \end{pmatrix} = \xi^0 \begin{pmatrix} F_{0i}^{CD} \\ D_0 \eta^E \end{pmatrix} = \xi^0 \zeta^l \begin{pmatrix} F_{li}^{CD} \\ D_l \eta^E \end{pmatrix} \quad (4.16)$$

which is a space diffeomorphism with parameter $\xi^0 \zeta^l$. Analogously, if further null eigenvectors are present, the time diffeomorphisms can be written as a linear combination containing also the generators of the corresponding symmetries.

5. Degrees of freedom count

We are finally in a position to compute the number of local degrees of freedom of the theory. If the only null eigenvectors of $\Omega_{\alpha\beta}$ are the three vectors $V_{\beta(l)}$ obtained above, the count goes as follows: one has 70 canonical variables $(A_i^{AB}, \eta^A, p_{AB}^i, \pi_A)$, 10 first class constraints G_{AB} associated with gauge invariance, 3 first class constraints H_i associated with spatial diffeomorphism invariance and $35 - 3 = 32$ second class constraints. The number \mathcal{N} of local degrees of freedom is therefore given by (see e.g [11])

$$\mathcal{N} = \frac{1}{2}(P - 2F - S) = 6$$

where P is the dimension of phase space and F and S are the number of first and second class constraints, respectively.

Of course this is the maximum possible number of degrees of freedom, which is reached if the matrix $\Omega_{\alpha\beta}$ has no further null eigenvectors besides the $V_{\beta(l)}$ and these are linearly independent. The validity of these conditions depends on the region of the phase space one is considering. For example, in the region corresponding to maximally symmetric spacetime ($F^{AB} = 0$) and vanishing η^A , the matrix $\Omega_{\alpha\beta}$ becomes null and hence all constraints are first class and no degrees of freedom are left. This is in accordance with the results found in [8] by a perturbative expansion.

In order to prove the existence of regions of the phase space where the number of degrees of freedom is maximal, one should check if there are explicit solutions of the constraints such that the conditions above are satisfied. Although, due to the complexity

of the matrix $\Omega_{\alpha\beta}$, we were not able to check this for explicit solutions, we find plausible that the conditions of maximality hold for generic solutions, since no accidental symmetries, which would give rise to further gauge invariances and hence to more null eigenvectors, seem to be present in the action.

To complete the hamiltonian analysis one should still compute the Dirac brackets , which permit one to get rid of the second class constraints. For two phase space functions A and B , these are given in general by

$$\{A, B\}^* = \{A, B\} - \int_{\Sigma} dz \{A, \phi_{\alpha}(z)\} J_{\alpha\beta}(z) \{\phi_{\beta}(z), B\}$$

where ϕ_{α} are the second class constraints and $J_{\alpha\beta}$ is the inverse of the matrix $\bar{\Omega}_{\alpha\beta}$ formed with the Poisson brackets $\{\phi_{\alpha}, \phi_{\beta}\}$. In general, for first order systems, one obtains non-trivial brackets between the fields [13,14]. We expect a similar situation to arise also in our case. However, we are not able to compute explicitly the Dirac brackets , since we lack an explicit expression for extracting the 32 independent second class constraints out of (4.1).

Once one has obtained the Dirac brackets , one can proceed to the quantization of the theory. However, due to the non-trivial structure of the constraint algebra, it still seems difficult to obtain a Hilbert space realization for it.

6. Conclusions

We have studied the hamiltonian dynamics of a gauge model with an action of topological form, which can be identified with a theory of gravity in four dimensions and generalizes some well-known two-dimensional models. We have shown that this model displays many similarities with the odd-dimensional Chern-Simons theories. The action is first order in the time derivatives and the hamiltonian analysis can be performed using the Dirac analysis of constrained systems. The theory admits first class constraints related to gauge and diffeomorphisms invariance, and a set of second class constraints. In particular, the generator of time diffeomorphisms is not independent from the other constraints. The computation of the local degrees of freedom shows that in contrast with the two-dimensional case, their number does not vanish. Unfortunately, it is not easy to explicitly separate the first class from the second class constraints and then to calculate

the Dirac brackets . This is quite disappointing in view of a possible quantization of the model.

It would be interesting to classify the local degrees of freedom in terms of their spin. This can be more easily achieved in a perturbative approach. Preliminary results indicate that in the riemannian limit (vanishing torsion), one has a spin-2 excitation (graviton) in the (anti)-de Sitter case, and two scalars in the Poincaré case. The remaining degrees of freedom of the full theory are of course due to the torsion.

Finally, we note that our investigations could easily be extended to higher dimensions and to different gauge groups, not directly related to gravitation.

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